Example of self-averaging in three dimensions: Anderson localization of electromagnetic waves in random distributions of pointlike scatterers

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A simple yet realistic theoretical model is used to study Anderson localization of electromagnetic waves in three-dimensional disordered dielectric media. The preliminary results presented in our previous paper [M. Rusek, A. Orłowski, and J. Mostowski, Phys. Rev. E **53**, 4122 (1996)] are substantially extended and a sound physical interpretation is proposed. Very striking universal properties of the spectra of random matrices describing the scattering from a collection of randomly distributed pointlike scatterers are discovered. The appearance of the band of localized electromagnetic waves, emerging in the limit of an infinite system, is numerically observed. [S1063-651X(97)03111-5]

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I. INTRODUCTION

Recently three-dimensional (3D) random dielectric structures with a typical length scale matching the wavelength of electromagnetic radiation have attracted a great deal of attention, both in the microwave and in the optical part of the spectrum. Propagation of electromagnetic waves in these structures resembles the properties of electrons in disordered semiconductors. Therefore, many ideas concerning transport properties of light and microwaves in such media exploit the theoretical methods and concepts of solid-state physics that have been developed over many decades. One of these concepts is electron localization in noncrystalline systems such as amorphous semiconductors or disordered insulators. As shown by Anderson [1], in a sufficiently disordered infinite material an entire *band* of electronic states can be spatially localized. In fact, the Anderson transition may be viewed as a transition from particlelike behavior described by the diffusion equation to wavelike behavior, which results in localization by interference. Indeed, the most plausible explanation of the Anderson localization is based on the interference effects in multiple elastic scattering [2].

As interference is the common property of all wave phenomena, the quest for some analogs of electron localization for other types of waves has been undertaken and many generalizations of electron localization exist, especially in the realm of electromagnetic waves [3–6]. So-called weak localization of electromagnetic waves manifesting itself as enhanced coherent backscattering is presently relatively well understood theoretically [7–9] and established experimentally [10–12]. The question is whether interference effects in 3D random dielectric media can reduce the diffusion constant to zero leading to strong localization. The crucial parameter is the mean free path l, which should be rather short [13–15]. It seems that a suspension of TiO_2 spheres in air is the system in which the shortest l values for visible light may be realized in practice. However, despite the observation of a scale dependence of the diffusion constant in such media, which may be considered as a reasonable indication of Anderson transition, there still is no convincing experimental demonstration that strong localization could be possible in 3D random dielectric structures.

A better understanding of the Anderson localization of electromagnetic waves requires sound theoretical models. Such models should be based directly on the Maxwell equations and they should be simple enough to provide calculations without too many approximations. In this paper we investigate a simple yet reasonably realistic coupled-dipole model describing the scattering of electromagnetic waves from a collection of randomly distributed pointlike dielectric particles. We restrict ourselves to the study of the properties of the stationary solutions $\vec{E}(\vec{r},t)=\mathrm{Re}[\vec{\mathcal{E}}(\vec{r})e^{-i\omega t}]$ of the Maxwell equations. Consequently, the polarization of the medium is considered to be the oscillatory function of time $\vec{P}(\vec{r},t)=\mathrm{Re}[\vec{\mathcal{P}}(\vec{r})e^{-i\omega t}]$. Calculating and analyzing spectra of certain random matrices, we observe numerically the appearance of the continuous *band* of localized electromagnetic waves. Consequences for Anderson localization of electromagnetic waves in 3D disordered dielectric media are discussed.

The main advantage of the presented approach is that we do not need to perform an average over the disorder. Generally speaking, there is a temptation to apply averaging procedures as soon as ''disorder'' is introduced into the model. Averaging of the scattered intensity over some random variable leads to a transport theory of localization [16–18]. But ''there is a very important and fundamental truth about random systems we must always keep in mind: no real atom is an average atom, nor is an experiment done on an ensemble of samples'' [19]. What we really need to properly understand the existing experimental results are probability distributions, not averages. Indeed, to perform any meaningful averaging procedure the assumption of infinite medium is needed. On the other hand, in all experiments we can study finite media only. Within our approach we can see how localization ''sets in'' for an increasing number of scatterers by studying the probability densities of eigenvalues of some random matrices.

This paper is organized as follows. In Sec. II we recall the point-scatterer approximation and analyze the basic ideas of the coupled-dipole model that serves as a theoretical tool in our investigations. We arrive at the system of linear equations determining the polarization of the medium for a given incident wave. In Sec. III eigenvalues of the random matrix corresponding to this set of equations are studied. Self-averaging of the lowest eigenvalue emerging in the limit of an infinite medium is discovered numerically. This phenomenon is illustrated graphically and observed features are compared with one-dimensional results. Note that in one dimension the possibility of self-averaging can be proved analytically. In Sec. IV a sound physical interpretation of the obtained results is proposed. Self-averaging of the lowest eigenvalue is considered as the signature of the appearance of the band of localized electromagnetic waves, emerging in the limit of infinite system. It can be understood as a counterpart of Anderson transition in solid-state physics. We finish with some comments and conclusions in Sec. V.

II. POINT-SCATTERER APPROXIMATION

Usually localization of light is studied experimentally in microstructures consisting of dielectric spheres with diameters and mutual distances being comparable to the wavelength [15]. On the other hand, the theory of multiple scattering of electromagnetic waves by dielectric particles is tremendously simplified in the limit of point scatterers. In principle, this approximation is justified only when the size of the scattering particles is much smaller than the wavelength. In practical calculations, however, many multiple-scattering effects can be obtained qualitatively for coupled electrical dipoles. Examples are universal conductance fluctuations [20], enhanced backscattering [21], and dependent scattering [22]. What really counts for localization is mainly the scattering cross section and not the bare size of the scatterer. Therefore, trying to understand the problem, we replace the dielectric spheres located at the points $\vec{r}_a$ by *single* electric dipoles

$$\vec{\mathcal{P}}(\vec{r})=\sum_a \vec{p}_a\delta(\vec{r}-\vec{r}_a), \tag{1}$$

with properly adjusted scattering properties.

To use safely the point dipole approximation it is essential to use a representation for the scatterers that fulfills the optical theorem rigorously and conserves energy in the scattering processes. These requirements give the following form of the coupling between the dipole moment and the electric field incident on the dipole [23]:

$$\frac{2}{3}ik^3\vec{p}_a=\frac{e^{i\phi}-1}{2}\vec{\mathcal{E}}'(\vec{r}_a), \tag{2}$$

where $k=\omega/c$ is the wave number in vacuum. To get some insight into the physical meaning of the parameter ϕ from Eq. (2) let us observe that it is directly related to the total scattering cross section σ of an individual dielectric sphere represented by the single dipole [23]:

$$k^2\sigma=\frac{3\pi}{2}(1-\cos\phi). \tag{3}$$

Therefore, ϕ is a function of frequency ω and physical parameters describing the spheres such as radius R and dielectric constant ϵ. Thus each choice of ϕ is in fact a choice of scatterers.

The field acting on the ath dipole

$$\vec{\mathcal{E}}'(\vec{r}_a)=\vec{\mathcal{E}}^{(0)}(\vec{r}_a)+\sum_{b\neq a}\vec{\nabla}\times\vec{\nabla}\times\vec{p}_b\frac{e^{ik|\vec{r}_a-\vec{r}_b|}}{|\vec{r}_a-\vec{r}_b|} \tag{4}$$

is the sum of some incident free field $\vec{\mathcal{E}}^{(0)}$, which obeys the Maxwell equations in vacuum, and waves scattered by all *other* dipoles. Now, inserting Eq. (2) into Eq. (4), it is easy to obtain the system of linear equations determining the field acting on each dipole $\vec{\mathcal{E}}'(\vec{r}_a)$ for a given incoming wave $\vec{\mathcal{E}}^{(0)}(\vec{r}_a)$ [23]:

$$\sum_b \overleftrightarrow{M}_{ab}\cdot\vec{\mathcal{E}}'(\vec{r}_b)=\vec{\mathcal{E}}^{(0)}(\vec{r}_a). \tag{5}$$

If we solve it and use again Eq. (2) to find $\vec{p}_a$, then we are able to find the electromagnetic field everywhere in space. A similar integral equation relating the stationary outgoing wave to the stationary incoming wave is known in the general scattering theory as the Lippmann-Schwinger equation [24]. A way of dealing with localized states in this formalism is to solve Eq. (5) as a homogeneous equation, i.e., for the incoming wave $\vec{\mathcal{E}}^{(0)}(\vec{r}_a)$ equal to zero [23].

III. SELF-AVERAGING

Perfectly localized waves exist only in *infinite* systems of dipoles [23]. To illustrate the appearance of the band of localized electromagnetic waves, emerging in the limit of infinite system, we have to study the properties of *finite* systems for an increasing number of dipoles N (while keeping the density constant). For each distribution of the dipoles $\vec{r}_a$ placed randomly inside a sphere with the uniform scaled density $n=1$ dipole per wavelength cubed we have diagonalized numerically the $\overleftrightarrow{M}$ matrix from Eq. (5) and obtained the lowest eigenvalue

$$\Lambda(\phi)=\min_j|\lambda_j(\phi)|. \tag{6}$$

The resulting probability distribution $P_\phi(\Lambda)$, calculated from different distributions of N dipoles, is normalized in the standard way $\int d\Lambda P_\phi(\Lambda)=1$. Let us now compare the surface plots of $P_\phi(\Lambda)$ (treated as a function of two variables ϕ and Λ) calculated for systems consisting of $N=100$ and 1000 dipoles. They are presented in Figs. 1 and 2, respectively. In addition, in Figs. 3 and 4 we provide corresponding contour plots. It is seen from inspection of all these plots that, for increasing system size [in our case it increased $(10)^{1/3}\simeq 2$ times], at some ϕ the probability distribution $P_\phi(\Lambda)$ apparently moves towards the $\Lambda=0$ axis and simultaneously its variance decreases. This tendency is easily seen, e.g., for values of $|\phi|$ that are close to π. Our numerical investigations indicate that in the limit of an infinite medium, the probability distribution $P_\phi(\Lambda)$ tends to the δ function

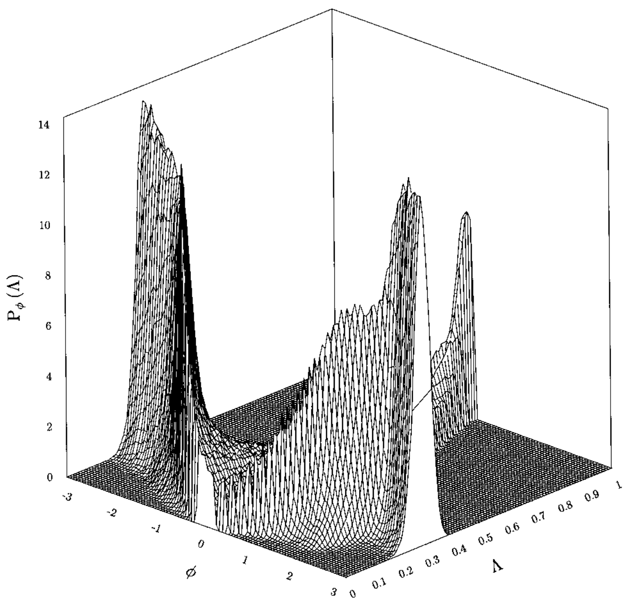

FIG. 1. Surface plot of the probability distribution $P_\phi(\Lambda)$ calculated for 10^4 systems of $N=100$ dipoles distributed randomly in a sphere with the uniform density $n=1$ sphere per wavelength cubed.

$$\lim_{N\to\infty} P_\phi(\Lambda)=\delta(\Lambda) \quad \text{for } |\phi|>\phi_{\rm cr}. \tag{7}$$

We have some numerical evidence that this fact is a general property of $\overleftrightarrow{M}$ matrices, not restricted to the considered case of one dipole per wavelength squared $n=1$ [although the parameter $\phi_{\rm cr}$ from Eq. (7) certainly may depend on n]. Of course we could justify Eq. (7) by a more orthodox approach based on a version of the finite-size scaling analysis that leads, however, to an analogous conclusion [26].

It follows from Eq. (7) that in the limit $N\to\infty$ the distribution function $P_\phi(\Lambda)$ has only one value for which it is nonzero. The quantity $\Lambda(\phi)$ at $|\phi|>\phi_{\rm cr}$ is then ''self-averaging'' and the random process has in fact become deterministic. Knowledge of the average then provides knowledge about ''almost every'' individual realization of the random system. This property implies that the average value applies to *every single* realization of the system, except for a few special ones (with measure zero). This means that for almost any random distribution of the dipoles $\vec{r}_a$, the equation $\lambda_j(\phi)=0$ holds. Therefore, the corresponding eigenvector $\vec{\mathcal{E}}'(\vec{r}_a)$ of the $\overleftrightarrow{M}$ matrix is a nonzero solution of the system of linear equations (5) for the incoming wave $\vec{\mathcal{E}}^{(0)}(\vec{r}_a)$ equal to zero. Thus a localized wave exists [23].

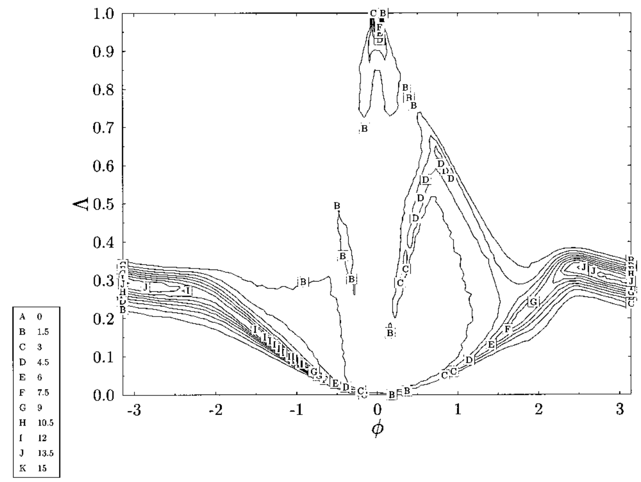

FIG. 3. Contour plot corresponding to Fig. 1.

In three dimensions, proofs of self-averaging are rare and in most cases quantities are not self-averaging [25]. For waves propagating in one-dimensional random systems (meaning that two out of three dimensions are translationally invariant and only the third is random) self-averaging can be demonstrated mathematically. For one-dimensional systems it was shown that for ''almost any'' energy or frequency an eigenfunction decays exponentially in space for almost any realization of the disorder [27,28]. This fact is also unambiguously confirmed within the one-dimensional version of our model. In Figs. 5 and 6 we present one-dimensional counterparts of Figs. 1 and 2. It is easily seen from inspection of these figures that Eq. (7) is satisfied also for systems

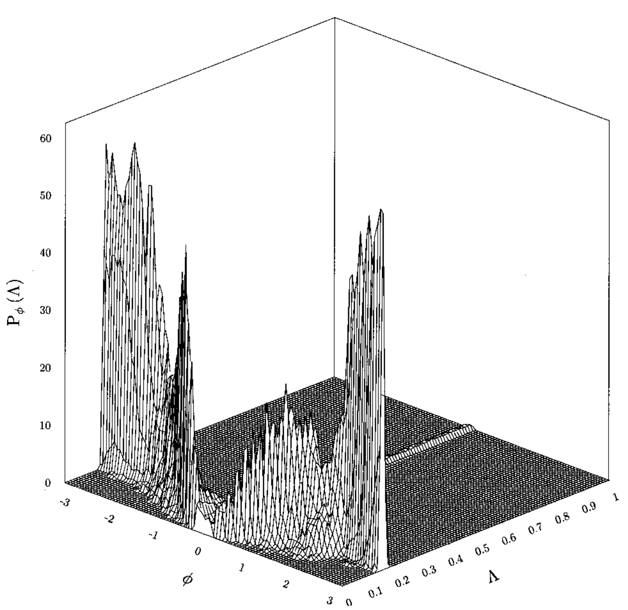

FIG. 2. Same as in Fig. 1, but for 10^3 systems of $N=1000$ dipoles.

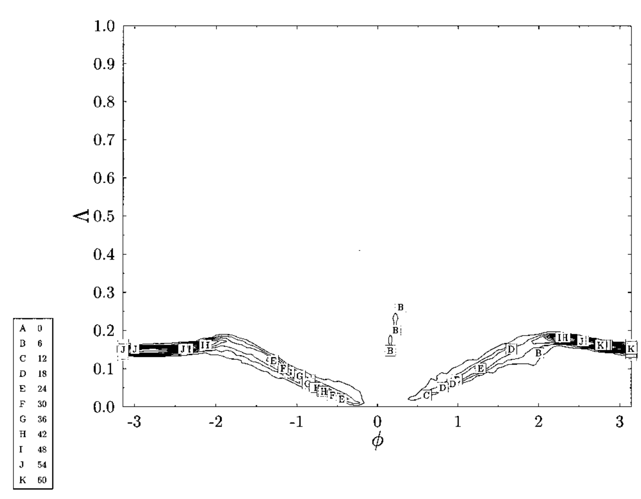

FIG. 4. Contour plot corresponding to Fig. 2.

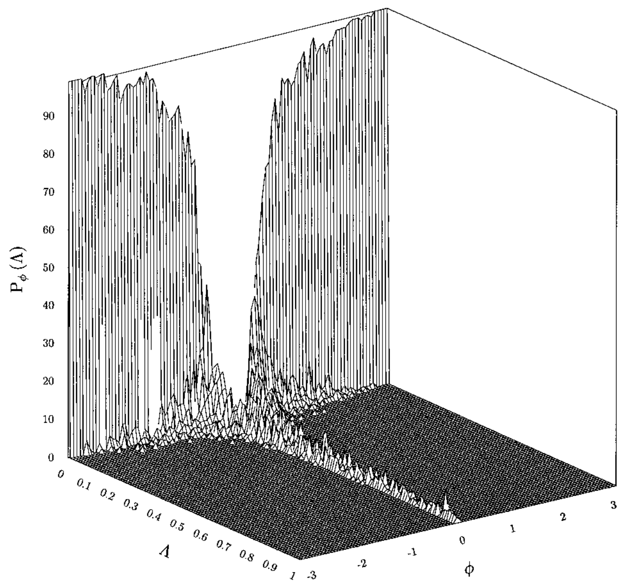

FIG. 5. Surface plot of the probability distribution $P_{\phi}(\Lambda)$ calculated for 10^2 systems of $N=100$ one-dimensional dipoles distributed randomly with the density $n=1$ slab per wavelength.

consisting of one-dimensional pointlike scatterers. Note that apparently $\phi_{\rm cr}=0$ in this case.

IV. ANDERSON LOCALIZATION

Electronic states in solids are usually either extended, by analogy with the Bloch picture for crystalline media, or localized around *isolated* spatial regions such as surfaces and impurities. However, in the case of a *sufficiently disordered* system a countable set of *discrete* energies corresponding to localized states becomes dense in some finite interval. But, physically speaking, it is impossible to distinguish between the allowed energies, which may be arbitrarily close to each other, and the spectrum is always a coarse-grained object.

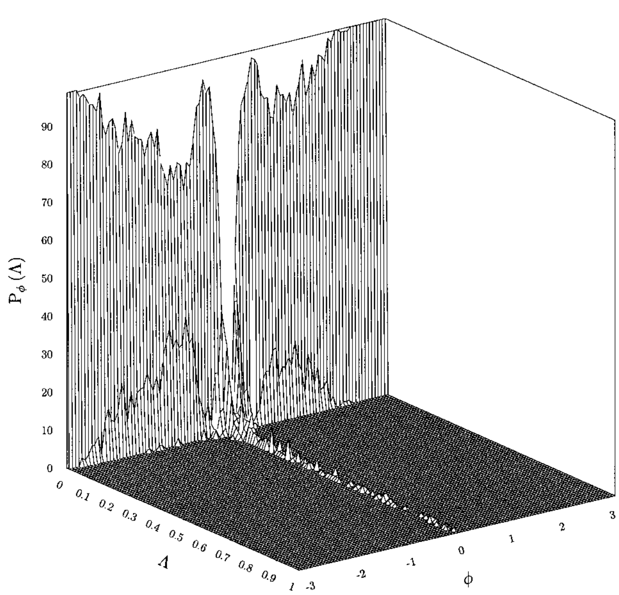

FIG. 6. Same as in Fig. 5, but for 10 systems of $N=1000$ one-dimensional dipoles.

Therefore, an entire *continuous band* of spatially localized electronic states appears. Anderson localization occurs when this happens [29].

Similarly, it is reasonable to expect that in the case of infinite and *random* collection of dielectric particles there can exist a band of localized electromagnetic waves corresponding to a continuous region of frequencies ω. This analogy allows us to elaborate a physical interpretation of the results obtained with the coupled-dipole model used. Let us now apply our model to a system of identical dielectric spheres with given radii R and dielectric constants $\epsilon(\omega)$ located randomly with uniform physical density η. First let us observe that in this case the parameter ϕ from Eq. (2) remains a function of the frequency, i.e., $\phi=\phi(\omega)$. On the other hand, as pointed out before, Eq. (7) holds not only for $n=1$ but for a whole range of n and therefore, for fixed η, for a range of frequencies ω. Thus the values of $\phi_{\rm cr}$ should be now regarded as functions of ω. Therefore, localized waves occur in almost any realization of the 3D random medium under consideration if $|\phi(\omega)|>\phi_{\rm cr}(\omega)$. This inequality determines a continuous region of frequencies ω corresponding to the band of localized waves. Indeed, after choosing a point from this region a localized wave of frequency (arbitrarily near) ω exists in almost any random distribution of the scatterers.

We see from Eqs. (3) and (7) that the total scattering cross section of individual particles σ must exceed some critical value $\sigma_{\rm cr}=\sigma(\phi_{\rm cr})$ before localization will take place in the limit $N\rightarrow\infty$. This fact is in perfect agreement with the scaling theory of localization [30]: In 3D random media a certain critical degree of disorder is needed for localization. Moreover, our preliminary calculations indicate that the value of $k^2\sigma_{\rm cr}$ may decrease with n, but *slower* than n^{-2}. Using the Rayleigh expression for the total scattering cross section σ of a dielectric sphere with radius R and dielectric constant ϵ [31],

$$k^2\sigma=\frac{8}{3\pi}(kR)^6\left|\frac{\epsilon-1}{\epsilon+1}\right|^2, \tag{8}$$

we conclude that in the long-wavelength limit the system of dielectric spheres distributed with constant density $\eta=k^3n/(2\pi)^3$ will be out of the localization regime. On the other hand, in the limit of small wavelengths, the propagation of light is ruled by the laws of geometrical optics and the point-scatterer approximation we use becomes invalid. Therefore, our results seem to agree with the common belief (see, e.g., [14,15]) that in three-dimensional media Anderson localization of light is possible only in a certain frequency window.

By analogy with the electron case, the phenomenon of Anderson localization of electromagnetic waves should manifest itself as an inhibition of the transmission in a spatially random dielectric medium. We have already some numerical evidence that it is actually true in the case of a one-dimensional system consisting of randomly distributed dielectric slabs. The validity of this connection in the considered three-dimensional model would attribute a sound interpretation and clear physical meaning to the continuous region of frequencies corresponding to localized waves. We expect that for each point ω from this region, incident waves with frequency ω will be totally reflected by almost any random distribution of the spheres $\vec{r}_a$ with scattering properties $\phi(\omega)$. This problem will be addressed in detail elsewhere.

V. SUMMARY

In this paper we have further developed and refined a quite realistic coupled-dipole model describing scattering of electromagnetic waves by a disordered dielectric medium. Its relative simplicity allowed us to discover some features of the Anderson localization of electromagnetic waves in 3D dielectric media without using any averaging procedures. Within our theoretical approach one can easily see how localization sets in for increasing system size. Very striking universal properties of the spectra of random matrices describing the scattering from a collection of randomly distributed pointlike scatterers have been observed. Self-averaging of the lowest eigenvalue emerging in the limit of an infinite medium has been discovered numerically. The appearance of the band of localized electromagnetic waves in three dimensions was demonstrated. A connection between this phenomenon and a dramatic inhibition of the propagation of electromagnetic waves in a spatially random dielectric medium has been sketched. It can be understood as a counterpart of Anderson transition in solid-state physics. Being aware of differences between electrons and photons, we discussed briefly the influence of the long-wavelength (Rayleigh) limit of elastic scattering of electromagnetic waves on results obtained within our model.

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